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Critical period bifurcations of a cubic system

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Abstract
We study bifurcations of the period function of a linear centre perturbed by third degree homogeneous polynomials. The approach is based on making use of algorithms of computational algebra.

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1. Introduction

Let $F(\rho,\lambda)$ ($\rho \in \mathbb{R}, \lambda = (\lambda_1,\ldots,\lambda_n) \in \mathcal{E} \subset \mathbb{R}^n$) be an analytic function. We write it near $\rho = 0$ in the form

$$\mathcal{F}(\rho,\lambda) = \sum_{k=0}^{\infty} f_k(\lambda)\rho^k \qquad (1)$$

where, for each integer $k \geqslant 0$, $f_k(\lambda)$ is an analytic function and the series (1) is convergent in a neighbourhood of $\rho = 0$. Denote by $n_{\lambda,\epsilon}$ the number of isolated zeros of $\mathcal{F}(\rho,\lambda)$ in the interval $0 < \rho < \epsilon$.

Definition 1. *Let $\lambda^* \in \mathcal{E}$ be such that $\mathcal{F}(\rho,\lambda^*) = 0$. We say that the* multiplicity *of $\mathcal{F}(\rho,\lambda)$ at λ^* with respect to the set $\mathcal{E}$ is equal to m if there exist $\delta_0 > 0$ and $\epsilon_0 > 0$, such that for every $0 < \epsilon < \epsilon_0$ and $0 < \delta < \delta_0$*

$$\max_{\lambda \in U_\delta(\lambda^*)\cap\mathcal{E}} n_{\lambda,\epsilon} = m.$$

Thus there arises the problem of finding a bound for the multiplicity of $\mathcal{F}(\rho,\lambda)$ at $\mathcal{E}$. There are two different cases:

(i) $f_0(\lambda^*) = \cdots = f_{s-1}(\lambda^*) = 0$, $f_s(\lambda^*) \neq 0$ for some $s \leqslant 0$
(ii) $f_i(\lambda^*) = 0$ for all $i = 0, 1, 2, \ldots$.

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In the first case, it is not difficult to see that the multiplicity of λ^* is less than or equal to s (see, e.g., [15]). Case (ii) is much more subtle, but there is a method for its treatment suggested by Bautin [2] (see also [4, 13, 15]).

In the case when $\mathcal{F}(\rho, \lambda)$ is the Poincaré return map and $\mathcal{E} = \mathbb{R}^{n^2+3n}$ is the space of parameters of a two-dimensional system of differential equations with polynomials of the degree n on the right-hand side (or some subspace of $\mathbb{R}^{n^2+3n}$), the problem of multiplicity is known as *the cyclicity problem* [2] and is also called *the local 16th Hilbert problem* [9, 11].

In the case when $\mathcal{F} = P(\rho) - 2\pi$, where $P(\rho)$ is the period function defined by formula (11) and $\mathcal{E}$ is the centre variety of system (4), we have the so-called *problem of bifurcations of critical periods*, which was considered for the first time by Chicone and Jacobs for quadratic systems and some Hamiltonian systems [4]. The bifurcation problem for small critical periods is analogous to the bifurcation problem of small amplitude limit cycles (cyclicity) when the parametric space is not $\mathbb{R}^{n^2+3n}$, but an affine variety in $\mathbb{R}^{n^2+3n}$.

In [4], the problem of bifurcations of critical periods is solved for the quadratic system. In the present paper, we investigate bifurcations of critical periods for the system with linear centre perturbed by homogeneous cubic polynomials, namely

$$\mathrm{i}\dot{x} = x - a_{20}x^3 - a_{11}x^2\bar{x} - a_{02}x\bar{x}^2 - a_{-13}\bar{x}^3. \tag{2}$$

The bifurcations of critical periods of (2) have been studied in [16]. Our method, which is based on the algorithms of computational algebra, differs from the methods used in [4, 16].

2. Preliminaries

Consider a real system

$$\dot{u} = -v + U(u, v) \qquad \dot{v} = u + V(u, v) \tag{3}$$

where $U(u, v)$ and $V(u, v)$ are polynomials without free and linear terms.

Using the complex variable $x = u + \mathrm{i}v$, we will write system (3) as a single equation

$$\mathrm{i}\frac{\mathrm{d}x}{\mathrm{d}t} = x - \sum_{(p,q)\in S} a_{pq}x^{p+1}\bar{x}^q \tag{4}$$

where a_{pq} are complex coefficients, $S = \{(m, k) : m + k \geqslant 1\}$ is a subset of $\{-1 \cup \mathbb{N}\} \times \mathbb{N}$ and $\mathbb{N}$ is the set of non-negative integers. Due to the form of its linear part, this system (equation) has either a centre or a focus at the origin in the real plane $\{(u, v) : x = u + \mathrm{i}v\}$. The origin of system (3) is *a centre* if all trajectories in its neighbourhood are closed and it is *an isochronous centre* if the period of oscillations is the same for all these trajectories. We also know from the Poincaré–Lyapunov theorem that (4) has a formal first integral of the form (6) (with y replaced by $\bar{x}$) if and only if the origin is a centre on the real plane u, v. In this case, the integral is analytic, and according to Vorob'ev's theorem [18, 1], the centre is isochronous if and only if system (3) is analytically linearizable.

In order to investigate the period function of equation (4), we consider instead of (4) the more general system

$$\mathrm{i}\frac{\mathrm{d}x}{\mathrm{d}t} = x - \sum_{(p,q)\in S} a_{pq}x^{p+1}y^q = X(x, y) \qquad -\mathrm{i}\frac{\mathrm{d}y}{\mathrm{d}t} = y - \sum_{(p,q)\in S} b_{qp}x^q y^{p+1} = -Y(x, y) \tag{5}$$

where x, y, a_{pq}, b_{qp} are complex variables. This system is equivalent to equation (4) in the case when $y = \overline{x}$, $b_{ij} = \overline{a}_{ji}$. We denote by (a, b) the vector of parameters of system (5),

$(a, b) = (a_{p_1q_1}, a_{p_2q_2}, \dots, a_{p_lq_l}, b_{q_lp_l}, \dots, b_{q_2p_2}, b_{q_1p_1}) \subset \mathbb{C}^{2l}$ and by $k[a, b]$ the polynomial ring in the variables a_{pq}, b_{qp} over a field k.

Following Dulac [8] and basing our definition on the Poincaré–Lyapunov theorem, we can introduce the notion of centre for system (5).

Definition 2. *We say that system (5) has a centre at the origin if there is a formal power series*

$$\Psi(x, y) = xy + \sum_{l+j=3}^{\infty} v_{l,j} x^l y^j \tag{6}$$

such that $\frac{\partial \Psi}{\partial x} X + \frac{\partial \Psi}{\partial y} Y \equiv 0$.

There are different algorithms to find necessary conditions for the existence of a centre and for linearizability (see, e.g., [5, 6, 14]). One of them is a transformation of system (5) to the normal form

$$\mathrm{i}\dot{x}_1 = x_1(1 + X_1(x_1, y_1)) \qquad -\mathrm{i}\dot{y}_1 = y_1(1 + Y_1(x_1, y_1)) \tag{7}$$

where $X_1(x_1, y_1) = \sum_{k=1}^{\infty} i_{kk}(x_1y_1)^k$ and $Y_1(x_1, y_1) = \sum_{k=1}^{\infty} j_{kk}(x_1y_1)^k$, by means of the change of coordinate

$$x = x_1 + \sum_{k+j\geqslant 2} h^{(1)}_{k-1,j} x_1^k y_1^j \qquad y = y_1 + \sum_{k+j\geqslant 2} h^{(2)}_{k,j-1} x_1^k y_1^j. \tag{8}$$

We call the polynomials i_{kk}, j_{kk} *the linearizability quantities* of system (5). The resonant coefficients $h^{(1)}_{k,k}, h^{(2)}_{k,k}$ of the series (8) can be chosen arbitrary, but we always set them equal to zero.

Let

$$G = \sum_{k=1}^{\infty} g_{kk}(x_1y_1)^k \qquad H = \sum_{k=1}^{\infty} p_{kk}(x_1y_1)^k$$

be the functions defined by the equations $G = (X_1 - Y_1)/2$, $H = (X_1 + Y_1)/2$. It follows from theorem 1 below that in the case when $y = \bar{x}$, the second equation is complex conjugate to the first one given by

$$\mathrm{i}\dot{x}_1 = x_1(1 + G(x_1\bar{x}_1) + H(x_1\bar{x}_1)) \tag{9}$$

where $G = \mathrm{i}\,\mathrm{Im}\,X_1$ and $H = \mathrm{Re}\,X_1$.

The next statement is a version of the Poincaré–Lyapunov theorem.

Proposition 1. *System (5) has a centre at the origin if and only if* $G(x_1y_1) \equiv 0$.

Proof. If $G \equiv 0$, then x_1y_1 is a first integral of (7). In this case, the transformation (8) is convergent [17, section 17], [3, sections 3 and 5] and, therefore, after the substitution inverse to (8), we get an analytical first integral of (5) of the form (6).

On the other hand, if system (5) has a first integral (6), then $G \equiv 0$ [3, section 6]. □

Let $x_1 = r\,\mathrm{e}^{-\mathrm{i}\theta}$ and suppose that the system has a centre, that is $G \equiv 0$. Then from (9), we obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + H(r^2). \tag{10}$$

The *period function*, $P(\rho)$, at ρ is defined to be the time by the closed orbit $r(\theta, \rho)$ while turning once around the origin. A centre is isochronous if $P(\rho)$ is constant. From (10), we get

$$P(\rho) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + H(\rho^2)} = \frac{2\pi}{1 + H(\rho^2)} = 2\pi \left(1 + \sum_{k=1}^{\infty} p_{2k}\rho^{2k}\right). \tag{11}$$

Given polynomials $f_1, \dots, f_s \in \mathbb{C}[a, b]$, we denote by $\langle f_1, \dots, f_s \rangle$ the ideal of $\mathbb{C}[a, b]$ generated by $f_1, \dots, f_s$ and by $\mathbf{V}(I)$ the (affine) variety of the ideal $I \subset \mathbb{C}[a, b]$,

$$\mathbf{V}(I) = \{(a, b) \in \mathbb{C}^{2l} : f(a, b) = 0, \forall f \in I\}.$$

It is easily seen that the polynomials p_{2k} are such that

$$p_2 = -p_{11} \qquad \text{and} \qquad p_{2k} \equiv -p_{kk} \bmod \langle p_{11}, \dots, p_{k-1,k-1} \rangle.$$

Obviously, the centre is isochronous if and only if $p_{2k} = 0$ for $k \geqslant 1$ or, equivalently, $p_{kk} = 0$ for $k \geqslant 1$. We call p_{mm} *the mth isochronicity quantity* and g_{mm} *the mth focus quantity*. They are polynomials of $\mathbb{C}[a, b]$. According to proposition 1, system (5) has a centre if and only if $g_{ii} = 0$ for all $i \geqslant 1$.

Definition 3. *The set* $V_{\mathcal{C}} = \mathbf{V}(\langle g_{11}, g_{22}, \dots, g_{ii}, \dots \rangle)$ *is called the centre variety of system (5) and the set* $V_{\mathcal{L}} = \mathbf{V}(\langle i_{11}, j_{11}, i_{22}, j_{22}, \dots, i_{kk}, j_{kk}, \dots \rangle)$ *is called the linearizability variety of system (5).*

Any system with coefficients from $V_{\mathcal{L}}$ is linearizable in a neighbourhood of the origin by a convergent substitution of the form (8).

Let $V \subset \mathbb{C}^{2l}$ be an affine variety. By $\mathbb{C}[V]$, we denote the ring of polynomial function $\psi : V \to \mathbb{C}$ on V. For any ideal $H = \langle h_1, \dots, h_s \rangle \subset \mathbb{C}[V]$, we define

$$\mathbf{V}_V(H) = \{(a, b) \in V : h(a, b) = 0 \text{ for all } h \in H\}.$$

$\mathbf{V}_V(H)$ is called *a subvariety* of V. It is easily seen that $\mathbf{V}_V(H)$ is contained in V and is an affine variety in $\mathbb{C}^{2l}$.

Consider the subvariety of the centre variety $V_{\mathcal{I}} = \mathbf{V}_{V_{\mathcal{C}}}(P)$, where $P = \langle p_{11}, p_{22}, \dots, p_{kk}, \dots \rangle \subset \mathbb{C}[V_{\mathcal{C}}]$ is the ideal of isochronicity quantities.

From the definitions of p_{mm} and $V_{\mathcal{C}}$ it follows.

Proposition 2. $V_{\mathcal{L}} = V_{\mathcal{I}}$.

For every point in $V_{\mathcal{C}}$, the corresponding system has a centre at the origin in the sense that there is a first integral of the form (6). However, if $(a, b) \in V_{\mathcal{C}}$ and $a_{pq} = \bar{b}_{qp}$ for all $(p, q) \in S$, then such a point corresponds to a real system of the form (4), which then has a topological centre at the origin in the plane $x = u + \mathrm{i}v$.

We will use the superscript R to denote the subsets of $V_{\mathcal{C}}$ and $V_{\mathcal{I}}$ corresponding to real systems; that is, $V_{\mathcal{C}}^R$ consists of those elements (a, b) in $V_{\mathcal{C}}$ such that $b = \bar{a}$, and similarly for $V_{\mathcal{I}}^R$.

To investigate the multiplicity (the critical period bifurcations) of the function

$$\mathcal{P}(a, b; \rho) = P(\rho) - 2\pi$$

we use Bautin's method, which is based on the following proposition.

Proposition 3. *Let*

$$\Psi(\theta, z) = f_1(\theta) z^{k_1}(1 + \psi_1(\theta, z)) + \cdots + f_j(\theta) z^{k_j}(1 + \psi_j(\theta, z)) \tag{12}$$

where $k_1 < \cdots < k_j, \theta = (\theta_1, \dots, \theta_s)$, *and* $f_i(\theta), \psi_i(\theta, z)$ *are series, convergent for* $|z| < \epsilon$, $\theta \in U_\delta(\theta^*)$ *and* $\psi_i(\theta, 0) = 0$ *for all* $i = 1, \dots, j$. *Then there exist* $0 < \epsilon_0 \leqslant \epsilon$ *and* $0 < \delta_0 \leqslant \delta$, *such that for each* $\theta \in U_{\delta_0}(\theta^*)$, *the equation with respect to* z $\Psi(\theta, z) = 0$ *has at most* $j - 1$ *isolated solutions in the neighbourhood* $0 < z < \epsilon_0$.

One can find the proof, for example, in [2, 13, 15].

Consider a function

$$\Psi(\theta, z) = \sum_{k=0}^{\infty} f_k(\theta) z^k \tag{13}$$

where $f_k(\theta) \in \mathbb{R}[\theta_1, \ldots, \theta_s]$ and at least one f_k is not zero. Let $I = \langle f_0, f_1, f_2, \ldots \rangle$ be the ideal generated by the polynomials f_j, $j = 0, 1, 2, \ldots$, and let $I_k = \langle f_0, \ldots, f_k \rangle$ be the ideal generated by the first $k + 1$ polynomials.

Denote by D_I the basis of I that is obtained by the following algorithm.

$D_I := \emptyset$; $I_{-1} := \langle 0 \rangle$; $k := 0$;
WHILE $I_{k-1} \neq I$ DO
  IF $I_k \neq I_{k-1}$ THEN $D_I := D_I \cup f_k$; $k := k + 1$.

Note that the algorithm produces an ascending chain of ideals $I_k \subset I_{k+m}$ and therefore it terminates, because the ring $\mathbb{R}[\theta_1, \ldots, \theta_s]$ is Noetherian.

The next statement follows from the results of [15].

Lemma 1. *Let $\Psi(\theta, z)$ be a series of the form (13) that is convergent for $|z| < \epsilon$ and $\theta \in U_\delta(\theta^*)$. If the basis D_I of the ideal I consists of n polynomials, $f_{k_1}, \ldots, f_{k_n}$, then there exist $0 < \epsilon_0 < \epsilon$ and $0 < \delta_0 < \delta$, such that the number of solutions of*

$$\Psi(\theta, z) = \sum_{k=0}^{\infty} f_k(\theta) z^k = 0$$

in the interval $0 < z < \epsilon_0$ is less than n for all $\theta \in U_{\delta_0}(\theta^*)$.

Corollary 1. *Assume that the coefficients $f_k(\theta)$ of the series (13) are polynomials with real coefficients and the ideal $I_m \subset \mathbb{C}[\theta]$ generated by the first m coefficients is radical. Denote by $\tilde{\Psi}(\theta, z) = \sum_{k=s+1}^{\infty} \tilde{f}_k(\theta) z^k$ the function $\Psi(\theta, z)|_{\mathbf{V}(I_m)}$, by $\tilde{I}$ the ideal of coefficients of $\tilde{\Psi}(\theta, z)$ in $\mathbb{C}[\mathbf{V}(I_m)]$ and assume that the basis $D_{\tilde{I}}$ of $\tilde{I}$ consists of t polynomials. Then the multiplicity of (real) $\Psi(\theta, z)$ is less than $m + t$.*

Proof. Denote by $[g]$ the equivalence class of g in $\mathbb{C}[\theta]/I_m$ and assume for simplicity that $D_{\tilde{I}} = \{[f_{m+1}], \ldots, [f_{m+t}]\}$. According to proposition 3 in order to prove the corollary, it is sufficient to show that the polynomials $f_1, \ldots, f_{m+t}$ form the basis of $I = \langle f_1, f_2, \ldots \rangle \subset \mathbb{R}[\theta]$. Because I_m is a radical ideal, we have $\mathbb{C}[\mathbf{V}(I_m)] \cong \mathbb{C}[\theta]/I_m$.

Consider a polynomial f_k with $k > m + t$. Then

$$[f_k] = \sum_{i=1}^{t} [\alpha_i][f_{m+i}] \quad \Longrightarrow \quad [f_k] = \left[\sum_{i=1}^{t} \alpha_i f_{m+i} \right]$$

yielding

$$f_k = \sum_{i=1}^{t} \alpha_i f_{m+i} + g \tag{14}$$

where $g \in I_m$. Taking into account that f_j are polynomials with real coefficients, we conclude that (14) holds also over $\mathbb{R}$, that is, $f_k \in I_{m+t} \subset \mathbb{R}[\theta]$. □

3. Properties of the linearizability quantities

With system (5), we associate the linear operator $L : \mathbb{N}^{2l} \to \mathbb{N}^2$,

$$L(\nu) = \begin{pmatrix} L^1(\nu) \\ L^2(\nu) \end{pmatrix} = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \nu_1 + \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} \nu_2 + \cdots + \begin{pmatrix} p_l \\ q_l \end{pmatrix} \nu_l$$
$$+ \begin{pmatrix} q_l \\ p_l \end{pmatrix} \nu_{l+1} + \cdots + \begin{pmatrix} q_2 \\ p_2 \end{pmatrix} \nu_{2l-1} + \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \nu_{2l} \tag{15}$$

where $(p_m, q_m) \in S$. Let $\mathcal{M}$ denote the set of all solutions $\nu = (\nu_1, \nu_2, \dots, \nu_{2l})$ with non-negative components of the equation

$$L(\nu) = \begin{pmatrix} k \\ k \end{pmatrix} \tag{16}$$

for all $k \in \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers. Obviously, $\mathcal{M}$ is an Abelian monoid.

Denote by $\mathbb{C}[\mathcal{M}]$ the subalgebra of $\mathbb{C}[a, b]$ generated by all monomials of the form

$$a_{p_1q_1}^{\nu_1} a_{p_2q_2}^{\nu_2} \cdots a_{p_lq_l}^{\nu_l} b_{q_lp_l}^{\nu_{l+1}} b_{q_{l-1}p_{l-1}}^{\nu_{l+2}} \cdots b_{q_1p_1}^{\nu_{2l}} \tag{17}$$

for all $\nu \in \mathcal{M}$. In order to simplify notation, we will abbreviate the monomial (17) with $\nu \in \mathbb{N}^{2l}$ by $\phi(\nu) = \phi((\nu_1, \dots, \nu_{2l}))$. For $\nu \in \mathbb{N}^{2l}$, let $\hat{\nu} = (\nu_{2l}, \nu_{2l-1}, \dots, \nu_1)$ be the involution of the vector ν. The result of applying the involution to the polynomial $f = \sum_{\nu\in\mathrm{supp}(f)} f_{(\nu)}\phi(\nu)$ is denoted by $\hat{f}$,

$$\hat{f} = \sum_{\nu\in\mathrm{supp}(f)} f_{(\nu)}\phi(\hat{\nu}).$$

The polynomial $g \in \mathbb{C}[a, b]$, $g = \sum_{\nu\in\mathrm{supp}(g)} g_{(\nu)}\phi(\nu)$ (with $\nu \in \mathbb{N}^{2l}$), is called an (m, n)-polynomial if for every $\nu \in \mathrm{supp}(g)$, the condition $L(\nu) = \binom{m}{n}$ holds.

Theorem 1.

(1) The coefficients $h_{kn}^{(1)}, h_{kn}^{(2)}$ of the transformation (18) are (k, n)-polynomials for all $(k, n) : k + n \geqslant 0, k, n \geqslant -1$. The linearizability quantities i_{kk} and j_{kk} belong to $\mathbb{Q}[\mathcal{M}]$ for all $k \geqslant 1$ and are (k, k)-polynomials.

(2) The coefficients of the transformation to normal form (8) and of the normal form (7) have the properties

$$h_{kl}^{(1)} = \hat{h}_{lk}^{(2)} \qquad i_{kk} = \hat{j}_{kk}.$$

The proof by induction is analogous to the proofs of theorems 1 and 2 in [14].

Define

$$\mathrm{Im}(\nu) := \phi(\nu) - \phi(\hat{\nu}) \qquad \mathrm{Re}(\nu) := \phi(\nu) + \phi(\hat{\nu}).$$

Corollary 2. *The isochronicity quantities of system (5) are of the form*

$$p_{kk} = \sum_{L(\nu)=(k,k)^t} p_{(\nu)} \,\mathrm{Re}(\nu).$$

Similar properties of the period constants were also obtained in [5]. But the definition of the period constants given there is different from our definition. So, the period constants and the isochronicity quantities can be different.

4. Critical periods of system (2)

Along with system (2), we consider the more general system

$$
\begin{aligned}
\mathrm{i}\dot{x} &= x - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2 - a_{-13}y^3 \\
-\mathrm{i}\dot{y} &= (y - b_{02}y^3 - b_{11}xy^2 - b_{20}x^2y - b_{3,-1}x^3).
\end{aligned}
\tag{18}
$$

The following results are known.

Theorem 2 ([6]). *The centre variety* $\mathbf{V}_{\mathcal{C}}$ *of system (18) consists of three irreducible components:*

$$\mathbf{V}_{\mathcal{C}} = \mathbf{V}(C_1) \cup \mathbf{V}(C_2) \cup \mathbf{V}(C_3) \tag{19}$$

where $C_1 = \langle a_{11} - b_{11}, 3a_{20} - b_{20}, 3b_{02} - a_{02}\rangle$, $C_2 = \langle a_{11}, b_{11}, a_{20} + 3b_{20}, b_{02} + 3a_{02}, a_{-13}b_{3,-1} - 4a_{02}b_{20}\rangle$ *and* $C_3 = \langle a_{20}^2 a_{-13} - b_{3,-1}b_{02}^2, a_{20}a_{02} - b_{20}b_{02}, a_{20}a_{-13}b_{20} - a_{02}b_{3,-1}b_{02}, a_{11} - b_{11}, a_{02}^2 b_{3,-1} - a_{-13}b_{20}^2\rangle$.

Systems from C_1 are Hamiltonian, systems from C_2 have a Darboux first integral and those from C_3 are reversible.

Theorem 3 ([6]). *The linearizability variety* $V_{\mathcal{L}}$ *of system (18) consists of the following irreducible components:*

(1) $b_{02} = a_{-13} = a_{02} = b_{11} = a_{11} = 0$,
(2) $b_{20} = b_{3,-1} = a_{20} = b_{11} = a_{11} = 0$,
(3) $112b_{20}^3 + 27b_{3,-1}^2 b_{02} = 49a_{-13}b_{20}^2 - 9b_{3,-1}b_{02}^2 = 21a_{-13}b_{3,-1} + 16b_{20}b_{02} = 343a_{-13}^2 b_{20} + 48b_{02}^3 = 7a_{02} + 3b_{02} = 3a_{20} + 7b_{20} = b_{11} = a_{11} = 0$,
(4) $b_{02} = b_{3,-1} = a_{02} = a_{20} + 3b_{20} = b_{11} = a_{11} = 0$,
(5) $b_{3,-1} = a_{-13} = a_{02} + b_{02} = a_{20} + b_{20} = b_{11} = a_{11} = 0$,
(6) $b_{20} = b_{3,-1} = a_{-13} = a_{02} = b_{11} = a_{11} = 0$,
(7) $b_{20} = a_{-13} = 3a_{02} + b_{02} = a_{20} = b_{11} = a_{11} = 0$.

We will denote by $J = \langle i_{11}, j_{11}, i_{22}, j_{22}, \ldots\rangle$ the ideal of all linearizability quantities and by $J_k = \langle i_{11}, j_{11}, \ldots, i_{kk}, j_{kk}\rangle$ the ideal of the first $2k$ linearizability quantities. Similarly, $P = \langle p_2, p_4, \ldots\rangle$ is the ideal of all isochronicity quantities and $P_k = \langle p_2, \ldots, p_{2k}\rangle$ is the ideal of the first k isochronicity quantities. Using a computer, we have found the normal form (7) of (5) up to $k = 6$. The expressions for $i_{11}, \ldots, i_{44}, j_{11}, \ldots, j_{44}$ are presented in the appendix. Since the expressions for $i_{55}, i_{66}, j_{55}, j_{66}$ are too long, we do not present them here. One can, however, easily compute them using any computer algebra system. We also have checked that $J_5 = J_6$ and $P_5 = P_6$ (the latter equality holds in $\mathbb{C}[V_{\mathcal{C}}]$).

Lemma 2. *For system (18)*

$$V_{\mathcal{L}} = \mathbf{V}(J_4) = \mathbf{V}_{V_{\mathcal{C}}}(P_4) = V_{\mathcal{I}} \tag{20}$$

where $J_4 = \langle i_{11}, j_{11}, i_{22}, j_{22}, \ldots, i_{44}, j_{44}\rangle$ *is an ideal in* $\mathbb{C}[a, b]$ *and* $P_4 = \langle p_2, p_4, p_6, p_8\rangle$ *is an ideal in* $\mathbb{C}[V_{\mathcal{C}}]$.

Proof. Using Singular [10], we computed the primary decomposition of J_4 and found that the associated primes are given by the polynomials defining the varieties (1)–(7) of theorem 3. It means that $V_{\mathcal{L}} = \mathbf{V}(J_4)$. Hence, taking into account proposition 2, we conclude that (20) holds. □

To investigate the bifurcations of critical periods, we use the following method. According to proposition 3, if we can represent the function $\mathcal{P}(a, b; \rho)$ in the form (12), then the multiplicity of $\mathcal{P}$ at any point is less than or equal to $j-1$. We are interested in bifurcations of critical periods for systems from $V_{\mathcal{I}}^R$; however, we consider this variety as enclosed in $V_{\mathcal{I}}$, and $V_{\mathcal{C}}^R$ as enclosed in $V_{\mathcal{C}}$, and we will look for representation (12) on the components of the centre variety $\mathbf{V}(C_i)$ (equivalently, for the basis of the ideal P in $\mathbb{C}(\mathbf{V}(C_i))$), where $\mathbf{V}(C_i)$ are the components defined in theorem 2. Because p_{kk} are polynomials with real coefficients, if such a representation exists in $\mathbb{C}[a,b]$, then it exists also in $\mathbb{R}[a,b]$.

In some cases, it is possible to get a bound for the number of bifurcating critical periods without finding a basis of the ideal of isochronicity quantities by using the following lemma.

Lemma 3 ([4]). *Assume that $\mathcal{F}(\rho,\lambda)$ is defined by (1) and each $f_k(\lambda)$, $k \geqslant 0$, is a homogeneous polynomial. If $f_0(\lambda) > 0$, when $\lambda \neq 0$, and if $deg(f_k) > deg(f_0)$ for $k > 0$, then there exist $\epsilon > 0$ and $\delta > 0$, such that for each $\lambda \in \mathbb{R}^n$ satisfying $0 < \lambda < \delta$, the equation $\mathcal{F}(\rho,\lambda) = 0$ has no solutions on $(0,\epsilon)$.*

We say that a perturbation of a system from $\mathbf{V}(C_i)$ is the proper perturbation if the perturbed system is also a system from $\mathbf{V}(C_i)$.

Lemma 4.

(i) The first four isochronicity quantities form a basis of P in $\mathbb{C}[\mathbf{V}(C_1)]$.
(ii) At most one critical period bifurcates from isochronous centres of the Hamiltonian system (2) (the component $\mathbf{V}(C_1)$ of the centre variety) after proper perturbations and this bound is sharp.

Proof. (i) On the component of Hamiltonian systems, we have $a_{11} = b_{11}$, $a_{20} = \frac{1}{3}b_{20}$ and $b_{02} = \frac{1}{3}a_{02}$. Due to corollary 1, we can assume that $p_2 = 0$ (yielding $a_{11} = b_{11} = 0$). Then we get

$$p_2 = 0 \qquad \tilde{p}_4 = (16a_{02}b_{20} + 9a_{-13}b_{3,-1})/12 \qquad \tilde{p}_6 = 5\big(a_{-13}b_{20}^2 + a_{02}^2 b_{3,-1}\big)/3$$
$$\tilde{p}_8 \equiv -\tfrac{105}{32}a_{-13}^2 b_{3,-1}^2 \bmod\langle p_2, p_3\rangle \qquad \tilde{p}_{10} \equiv 0 \bmod\langle p_2, p_3\rangle \tag{21}$$

where $\tilde{p}_{2k} = p_{2k}|_{b_{11}=0}$.

To prove (i) it is sufficient to show that for $k \geqslant 6 \tilde{p}_{2k} \in \langle \tilde{p}_4, \tilde{p}_6, \tilde{p}_8\rangle$.

Note that $\mathcal{H} = \{(2,0,0,1,0,0,0,0)$, $(0,0,0,0,1,0,0,2)$, $(1,1,0,0,0,0,0,0)$, $(0,0,0,0,0,0,1,1)$, $(1,0,0,1,0,1,0,0)$, $(0,0,1,0,1,0,0,1)$, $(0,0,2,0,1,0,0,0)$, $(0,0,0,1,0,2,0,0)$, $(0,1,0,0,0,0,0,0)$, $(0,0,0,0,0,0,1,0)$, $(1,0,0,0,0,0,0,1)$, $(0,0,1,0,0,1,0,0)$, $(0,0,0,1,1,0,0,0)\}$ is the Hilbert basis of the monoid $\mathcal{M}$ of system (18) [12]. Here and below we assume the order $a_{20} > a_{11} > a_{02} > a_{-13} > b_{3,-1} > b_{20} > b_{11} > b_{02}$. So, for instance, we have that $\phi((2,0,0,1,0,0,0,0)) = a_{20}^2 a_{-13}$.

One can check that

$$\big(a_{-13}b_{20}^2 - a_{02}^2 b_{3,-1}\big)^2 = (\mathrm{Im}((0,0,2,0,1,0,0,0)))^2 \equiv 0 \bmod\langle \tilde{p}_4, \tilde{p}_6, \tilde{p}_8\rangle \tag{22}$$

$$a_{02}^2 b_{20}^2 \equiv 0 \bmod\langle \tilde{p}_4, \tilde{p}_6, \tilde{p}_8\rangle \qquad a_{02}b_{20}a_{-13}b_{3,-1} \equiv 0 \bmod\langle \tilde{p}_4, \tilde{p}_6, \tilde{p}_8\rangle. \tag{23}$$

Using $\mathcal{H}$, it is easy to see that if $\mathrm{Re}(\theta)$ is a term of a polynomial p_{2k} with $k \geqslant 6$, then θ is of the form $\theta = \zeta((0,0,2m,0,m,0,0,0))$ with $\zeta = (a_{02}b_{20})^{\alpha}(a_{-13}b_{3,-1})^{\beta}$, where either $m \geqslant 2$ or $\alpha + \beta \geqslant 2$.

In the first case using (21), (22), the expression for $\tilde{p}_6$ given above and the formulae

$$\mathrm{Im}(\nu + \mu) = \tfrac{1}{2}\,\mathrm{Im}(\nu)\mathrm{Re}(\mu) + \tfrac{1}{2}\,\mathrm{Re}(\mu)\mathrm{Im}(\nu)$$
$$\mathrm{Re}(\nu + \mu) = \tfrac{1}{2}\,\mathrm{Im}(\nu)\mathrm{Im}(\mu) + \tfrac{1}{2}\,\mathrm{Re}(\mu)\mathrm{Re}(\nu)$$

we conclude that

$$\mathrm{Re}(\theta) \equiv 0 \bmod \langle p_4, p_6, p_8 \rangle. \tag{24}$$

In the second case, (24) follows immediately from (21) and (23). Thus the statement (i) is proved.

(ii) It follows from (i) that at most three critical periods can bifurcate from isochronous centres of Hamiltonian systems. However, the bound can be improved using lemma 3. Indeed, observing that p_4 is negatively defined (because $b_{20} = \bar{a}_{02}$, $b_{3,-1} = \bar{a}_{-13}$ for system (2)) and that due to corollary 2 each p_{2k} is a homogeneous polynomial, we conclude from lemma 3 and corollary 1 that at most one critical period bifurcates from isochronous centres of Hamiltonian systems (2) after proper perturbations. □

Lemma 5. *No critical periods of system (2) bifurcate from isochronous centres of the component* $\mathbf{V}(C_2)$ *of the centre variety after proper perturbations.*

Proof. In this case, $p_2 \equiv 0$ and $p_4|_{\{a_{20}=-3b_{20}, b_{02}=-3a_{02}\}} = (3a_{-13}b_{3,-1} - 8a_{02}b_{20})/4 \neq 0$ on C_2, because one of the defining equations of C_2 is $a_{-13}b_{3,-1} - 4a_{02}b_{20}$. Therefore, no bifurcation is possible on C_2. □

To investigate the remaining component $\mathbf{V}(C_3)$, we need

Lemma 6. *Let system a^* correspond to the point of the centre variety in the intersections of two sets, A and B, of the variety and assume that there are parametrizations of A and B in a neighbourhood of a^*, such that the function $\mathcal{P}(\rho) = P(\rho) - 2\pi$ can be written in the form (12) with k_l equal to m_A and m_B, respectively. Then, the multiplicity of $P(\rho)$ with respect to $A \cup B$ is equal to* $\max(m_A, m_B)$.

Proof. Apply proposition 3 to each of the functions $\mathcal{P}(\rho)$ at A and B. □

Lemma 7. *At most three critical periods of system (2) bifurcate from isochronous centres of the component* $\mathbf{V}(C_3)$ *after proper perturbations, and this bound is sharp.*

Proof. Using the implicitization algorithm for polynomial parametrization (see, e.g., [7, chapter 3]), one can see that

$$\begin{array}{llll} a_{11} = b_{11} = u & a_{20} = sw & b_{20} = w & b_{02} = sv \\ a_{02} = v & a_{-13} = tv^2 & b_{3,-1} = tw^2 & \end{array} \tag{25}$$

is a polynomial parametrization of $\mathbf{V}(C_3)$. Indeed, computing a Gröbner basis of the ideal

$$\langle a_{11} - u, b_{11} - u, a_{20} - sw, b_{02} - sv, a_{-13} - tv^2, b_{3,-1} - tw^2, a_{02} - v, b_{20} - w \rangle \tag{26}$$

with respect to lex with $t > s > v > w > u > a_{11} > b_{11} > a_{20} > b_{02} > a_{02} > b_{20} > a_{-13} > b_{3,-1}$, we get that it is

$$\begin{aligned} \{ & a_{-13}b_{20}^2 - a_{02}^2 b_{3,-1}, -a_{-13}a_{20}b_{20} + a_{02}b_{02}b_{3,-1}, -a_{02}a_{20} + b_{02}b_{20}, -a_{-13}a_{20}^2 + b_{02}^2 b_{3,-1}, \\ & - a_{11} + b_{11}, b_{11} - u, b_{20} - w, a_{02} - v, -a_{20} + b_{20}s, -b_{02} + a_{02}s, b_{3,-1} \\ & - b_{20}^2 t, a_{-13} - a_{02}^2 t, -a_{-13}s + a_{02}b_{02}t, a_{-13}s^2 - b_{02}^2 t, \\ & - b_{3,-1}s + a_{20}b_{20}t, b_{3,-1}s^2 - a_{20}^2 t \}. \end{aligned} \tag{27}$$

Polynomials of the basis which do not depend on t, s, v, w, u are exactly the generating polynomials of the ideal C_3. Due to the implicitization algorithm, this means that (25) is a polynomial parametrization of $\mathbf{V}(C_3)$.

After the substitution (25), we get

$$p_4 = \tfrac{1}{4}vw(4 + 4s + 3vwt^2) \qquad \tilde{\tilde{p}}_6 = p_6|_{u=0}/(vw) = vwt(3 + s)(7 + 3s)/8$$
$$\tilde{\tilde{p}}_8 = p_8|_{u=0}/(vw) \equiv \tfrac{5}{36}vw(1 + s)(7 + 3s) \operatorname{mod}\langle p_2, p_4\rangle$$

(to get $\tilde{\tilde{p}}_8$ we have used lex with $v > w > s > t$). Direct calculations show that $p_{10} \in \langle p_2, p_4, p_6, p_8\rangle$. Computing (e.g., with Singular [10]) one can also see that $\langle p_2, p_4, \tilde{\tilde{p}}_6, \tilde{\tilde{p}}_8\rangle$ is a radical ideal and $\langle p_2, p_4, \tilde{\tilde{p}}_6, \tilde{\tilde{p}}_8\rangle = \langle u, p_4, \tilde{\tilde{p}}_6, \tilde{\tilde{p}}_8\rangle = \sqrt{\langle p_2, p_4, p_6, p_8\rangle}$ (where $\sqrt{I}$ stands for the radical of the ideal I). Therefore, in order to prove that $\tilde{p}_{2k} = p_{2k}|_{u=0} \in \langle p_4, \tilde{p}_6, \tilde{p}_8\rangle$ $(k \geqslant 6)$, it is sufficient to show that it is of the form

$$\tilde{p}_{2k} = v^2w^2 f_k(s, t, v, w). \tag{28}$$

Indeed, if (28) holds then

$$\tilde{p}_{2k}|_{\mathbf{V}(\langle p_4, \tilde{p}_6, \tilde{p}_8\rangle)} \equiv 0 \quad \Longleftrightarrow \quad vwf_k|_{\mathbf{V}(\langle p_4, \tilde{\tilde{p}}_6, \tilde{\tilde{p}}_8\rangle)} \equiv 0.$$

Taking into account that $\langle p_4, \tilde{\tilde{p}}_6, \tilde{\tilde{p}}_8\rangle$ is a radical ideal and using lemma 2, we get that $vwf_k = p_4h_1 + \tilde{\tilde{p}}_6h_2 + \tilde{\tilde{p}}_8h_3$, yielding $\tilde{p}_{2k} = vwp_4h_1 + \tilde{p}_6h_2 + \tilde{p}_8h_3$. This implies that $p_{2k} \in \langle p_2, p_4, p_6, p_8\rangle$.

In order to see that (28) holds, we note that after the substitutions (25) with $u = 0$ for every element $\mu \in \mathcal{H}$ the monomial $\phi(\mu)$ contains the multiple vw. Every exponent of any monomial of p_{2k} with $k \geqslant 6$ is a sum of at least two basis monomials. Therefore, $p_{2k}|_{u=0}$ has the form (28) for all $k \geqslant 6$.

However, probably, the parametrization (25) does not fill up all of its variety $\mathbf{V}(C_3)$. Using the extension theorem [7, chapter 3], we see from (27) that if $a_{02} \neq 0$ and $b_{20} \neq 0$ then any partial solution (i.e., a solution to $a_{-13}b_{20}^2 - a_{02}^2b_{3,-1} = -a_{-13}a_{20}b_{20} + a_{02}b_{02}b_{3,-1} = -a_{02}a_{20} + b_{02}b_{20} = -a_{-13}a_{20}^2 + b_{02}^2b_{3,-1} = -a_{11} + b_{11} = 0$) can be extended to the variety of the ideal (26). However, if $a_{02}b_{20} = 0$, then the extension theorem does not guarantee this. Therefore, we do not know whether the parametrization fills up the entire variety.

Using the other parametrization

$$\begin{aligned} &a_{11} = b_{11} = u \qquad b_{20} = sv \qquad a_{20} = v \qquad a_{02} = sw \\ &b_{02} = w \qquad a_{-13} = tw^2 \qquad b_{3,-1} = tv^2 \end{aligned} \tag{29}$$

we get

$$p_4 = vw(4s + 4s^2 + 3vwt^2)/4 \qquad \tilde{\tilde{p}}_6 = p_6|_{u=0}/(vw) = vwt(1 + 3s)(7s + 3)/8$$
$$\tilde{\tilde{p}}_8 = p_8|_{u=0}/(vw) \equiv -\tfrac{5}{108}svw(1 + s)(7s + 3) \operatorname{mod}\langle p_4, p_6\rangle.$$

The ideal $\langle p_4, \tilde{\tilde{p}}_6, \tilde{\tilde{p}}_8\rangle$ is a radical ideal in the ring $\mathbb{C}[w, v, s, t]$; therefore, similarly as above, we conclude that $p_{2k} \in \langle p_2, p_4, p_6, p_8\rangle$ for $k > 4$. In this case, any partial solution with $a_{20} \neq 0$ can be extended to the entire variety.

It remains to consider the points of $\mathbf{V}(C_3)$ where $a_{20} = a_{02} = 0$, which are parametrized by

$$\begin{aligned} &a_{11} = b_{11} = u \qquad b_{20} = 0 \qquad a_{20} = 0 \qquad a_{02} = 0 \\ &b_{02} = 0 \qquad a_{-13} = w \qquad b_{3,-1} = v. \end{aligned} \tag{30}$$

On this set $p_2 = u$, $p_4|_{u=0} = \frac{3}{4}wv$ and corollary 2 yields that $P = \langle p_2, p_4\rangle$.

Let now a^* be a point of $\mathbf{V}(C_3)$ and $U \subset \mathbf{V}(C_3)$ be a small neighbourhood of a^*. Obviously, U is covered by the union of parametrizations (25), (29) and (30). Therefore, using lemma 6, we conclude that at most three small critical periods can appear after proper perturbations of system a^*.

From the above expressions for p_2, $\tilde{p}_4$, $\tilde{\tilde{p}}_6$, $\tilde{\tilde{p}}_8$, it is easily seen that there are values of parameters such that the function $\mathcal{P}(\rho)$ has three small zeros (or see [16] for details). □

From lemmas 4–7, we obtain

Theorem 4. *At most three critical periods bifurcate from isochronous centres of system (2) and there are systems of the form (2) with three small critical periods.*

To conclude, we have presented an approach to the investigation of small bifurcations of critical periods of polynomial systems and applied it to system (2). Our results for (2) agree with those obtained earlier in [16] by another method. In our opinion, the main problem in this area is to develop an *algorithmic* approach to precisely locate and count the 'small' zeros of the period functions and the 'small' fixed points of Poincaré maps.

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Appendix

We present here the coefficients i_{kk}, j_{kk} of the normal form (7) of system (18):

$$i_{11} = a_{11} \qquad j_{11} = b_{11}$$

$$i_{22} = (-4a_{02}a_{20} - 4a_{02}b_{20} - 3a_{-13}b_{3,-1})/4$$

$$j_{22} = (-4a_{02}b_{20} - 4b_{02}b_{20} - 3a_{-13}b_{3,-1})/4$$

$$\begin{aligned} i_{33} = \big(& -6a_{-13}a_{20}^2 + 4a_{11}a_{20}b_{02} - 16a_{02}a_{20}b_{11} - 20a_{02}a_{11}b_{20} - 24a_{-13}a_{20}b_{20} - 8a_{02}b_{11}b_{20} \\ & - 18a_{-13}b_{20}^2 - 24a_{02}^2b_{3,-1} - 11a_{11}a_{-13}b_{3,-1} \\ & - 8a_{02}b_{02}b_{3,-1} - 6a_{-13}b_{11}b_{3,-1}\big)/16 \end{aligned}$$

$$\begin{aligned} j_{33} = \big(& 4a_{20}b_{02}b_{11} - 8a_{02}a_{11}b_{20} - 8a_{-13}a_{20}b_{20} - 16a_{11}b_{02}b_{20} - 20a_{02}b_{11}b_{20} - 24a_{-13}b_{20}^2 \\ & - 18a_{02}^2b_{3,-1} - 6a_{11}a_{-13}b_{3,-1} - 24a_{02}b_{02}b_{3,-1} \\ & - 6b_{02}^2b_{3,-1} - 11a_{-13}b_{11}b_{3,-1}\big)/16 \end{aligned}$$

$$\begin{aligned} i_{44} = \big(& 144a_{02}a_{11}^2a_{20} + 144a_{11}a_{-13}a_{20}^2 - 96a_{11}^2a_{20}b_{02} + 96a_{02}a_{11}a_{20}b_{11} - 216a_{-13}a_{20}^2b_{11} \\ & + 240a_{11}a_{20}b_{02}b_{11} - 432a_{02}a_{20}b_{11}^2 - 288a_{02}a_{11}^2b_{20} - 288a_{02}^2a_{20}b_{20} \\ & - 72a_{11}a_{-13}a_{20}b_{20} - 192a_{02}a_{20}b_{02}b_{20} - 240a_{02}a_{11}b_{11}b_{20} \\ & - 576a_{-13}a_{20}b_{11}b_{20} - 192a_{02}^2b_{20}^2 - 516a_{11}a_{-13}b_{20}^2 - 96a_{02}b_{02}b_{20}^2 \\ & - 234a_{-13}b_{11}b_{20}^2 - 582a_{02}^2a_{11}b_{3,-1} - 132a_{11}^2a_{-13}b_{3,-1} - 660a_{02}a_{-13}a_{20}b_{3,-1} \\ & - 192a_{02}a_{11}b_{02}b_{3,-1} - 144a_{-13}a_{20}b_{02}b_{3,-1} - 336a_{02}^2b_{11}b_{3,-1} \\ & - 120a_{11}a_{-13}b_{11}b_{3,-1} + 24a_{02}b_{02}b_{11}b_{3,-1} - 18a_{-13}b_{11}^2b_{3,-1} \\ & - 1120a_{02}a_{-13}b_{20}b_{3,-1} - 300a_{-13}b_{02}b_{20}b_{3,-1} - 81a_{-13}^2b_{3,-1}^2\big)/192 \end{aligned}$$

$$\begin{aligned} j_{44} = \big(& 240a_{11}a_{20}b_{02}b_{11} - 96a_{20}b_{02}b_{11}^2 - 96a_{02}^2a_{20}b_{20} + 24a_{11}a_{-13}a_{20}b_{20} - 432a_{11}^2b_{02}b_{20} \\ & - 192a_{02}a_{20}b_{02}b_{20} - 240a_{02}a_{11}b_{11}b_{20} - 192a_{-13}a_{20}b_{11}b_{20} + 96a_{11}b_{02}b_{11}b_{20} \end{aligned}$$

$$
\begin{aligned}
&- 288a_{02}b_{11}^2b_{20} + 144b_{02}b_{11}^2b_{20} - 192a_{02}^2b_{20}^2 - 336a_{11}a_{-13}b_{20}^2 - 288a_{02}b_{02}b_{20}^2 \\
&- 582a_{-13}b_{11}b_{20}^2 - 234a_{02}^2a_{11}b_{3,-1} - 18a_{11}^2a_{-13}b_{3,-1} - 300a_{02}a_{-13}a_{20}b_{3,-1} \\
&- 576a_{02}a_{11}b_{02}b_{3,-1} - 144a_{-13}a_{20}b_{02}b_{3,-1} - 216a_{11}b_{02}^2b_{3,-1} \\
&- 516a_{02}^2b_{11}b_{3,-1} - 120a_{11}a_{-13}b_{11}b_{3,-1} - 72a_{02}b_{02}b_{11}b_{3,-1} \\
&+ 144b_{02}^2b_{11}b_{3,-1} - 132a_{-13}b_{11}^2b_{3,-1} - 1120a_{02}a_{-13}b_{20}b_{3,-1} \\
&- 660a_{-13}b_{02}b_{20}b_{3,-1} - 81a_{-13}^2b_{3,-1}^2\big)\big/192.
\end{aligned}
$$

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